

DISCRETIZATION IN GENERALIZED COORBIT SPACES: EXTENSIONS, ANNOTATIONS AND ERRATA FOR “CONTINUOUS FRAMES, FUNCTION SPACES AND THE DISCRETIZATION PROBLEM” BY M. FORNASIER AND H. RAUHUT

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ABSTRACT. During the process of writing the manuscript [9], the work [4] by Fornasier and Rauhut was one of the major foundations of our results and, naturally, we found ourselves going back to reading that contribution once and again.

In particular in Section 5, which is concerned with the discretization problem, we have found some typographical errors, small inaccuracies and some parts that we just would have wished to be slightly more accessible. Finally, for our own theory, a generalization of a central definition required us to verify that all the derivations in [4, Section 5] still hold after the necessary modifications.

Considering the importance of the results in [4] for the community, this was reason enough to start this side project of re-writing that least accessible portion of Fornasier and Rauhut’s manuscript, eliminating the errors we found, adding annotations where we consider them useful and modifying the results to take into account the generalization we require for our own work.

What you see is the result of our endeavor, a one-to-one substitute for [4, Section 5].

1. DISCLAIMER

The manuscript at hand is no complete scientific paper, but supposed to replace Section 5 “Discrete Frames” in [4], an open access version of which is available at <http://arxiv.org/abs/math/0410571>. We therefore use the notation and results in Sections 1-4 therein without re-introducing them. Where applicable, the text in [4] will be reproduced 1:1, extended where it was deemed useful and modified where necessary. Changes and additions are contained in blue, except for minor efforts to make notation more consistent. Major changes are justified in annotations. For definitions, results, equations and so on, we use the enumeration scheme from the arXiv version of [4], more precisely <http://arxiv.org/abs/math/0410571v1>.

If you find this manuscript useful for your own work, we kindly ask you to cite our associated contribution [9], to which this manuscript can be considered an online addendum. If you happen to find any oversights of our own or you have any suggestions for improvement, please contact us through the e-mail address provided on the last page of this manuscript.

5. DISCRETE FRAMES

In this section we investigate conditions under which one can extract a discrete frame from the continuous one. In particular, we will derive atomic decompositions and *Banach frames* for the associated coorbit spaces.

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The basic idea is to cover the index set X by some suitable covering $\mathcal{U} = (U_i)_{i \in I}$ with countable index set I such that the kernel R does not “vary too much” on each set U_i . This variation is measured by an auxiliary kernel¹ $\text{osc}_{\mathcal{U}, \Gamma}(x, y)$ associated to R . Choosing points $x_i \in U_i$, $i \in I$, we obtain a sampling of the continuous frame $\{\psi_x\}_{x \in X}$. Under certain conditions on $\text{osc}_{\mathcal{U}, \Gamma}$ the sampled system $\{\psi_{x_i}\}_{i \in I}$ is indeed a frame for \mathcal{H} , [respectively a Banach frame \(atomic decomposition\) for CoY \(CoY\)](#).

We start with a definition.

Definition 5.1. A family $\mathcal{U} = (U_i)_{i \in I}$ of subsets of X is called *(discrete) admissible covering* of X if the following conditions are satisfied.

- Each set U_i , $i \in I$ is relatively compact and has non-void interior.
- It holds $X = \bigcup_{i \in I} U_i$.
- There exists some constant $N > 0$ such that

$$(5.1) \quad \sup_{j \in I} \#\{i \in I, U_i \cap U_j \neq \emptyset\} \leq N < \infty.$$

Furthermore, we say that an admissible covering $\mathcal{U} = (U_i)_{i \in I}$ is *moderate* if it fulfills the following additional conditions.

- There exists some constant $D > 0$ such that $\mu(U_i) \geq D$ for all $i \in I$.
- There exists a constant \tilde{C} such that

$$(5.2) \quad \mu(U_i) \leq \tilde{C} \mu(U_j) \quad \text{for all } i, j \text{ with } U_i \cap U_j \neq \emptyset.$$

Note that the index set I is countable because X is σ -compact. We remark further that we do not require the size of the sets U_i (measured with μ) to be bounded from above. We only require a lower bound. Condition (5.2) means that the sequence $(\mu(U_i))_{i \in I}$ is \mathcal{U} -moderate in the sense of [2, Definition 3.1]. If the sets U_i do not overlap at all, i.e. they form a partition, then this condition is satisfied trivially. A recipe for the construction of more general admissible coverings with property (5.2) is discussed in [1] together with some relevant examples.

For the aim of discretization we have to restrict the class of admissible weight functions (resp. the class of function spaces Y). From now on we require that there exists a moderate admissible covering $\mathcal{U} = (U_i)_{i \in I}$ of X and a constant $C_{m, \mathcal{U}}$ such that

$$(5.3) \quad \sup_{x, y \in U_i} m(x, y) \leq C_{m, \mathcal{U}} \quad \text{for all } i \in I.$$

Of course, the trivial weight 1 has this property (provided of course that moderate admissible coverings exist), so that unweighted $L^p(X)$ -spaces are admitted. Moreover, if w is a continuous weight on X , then property (5.3) of its associated weight on $X \times X$ defined by (3.6) means that w is \mathcal{U} -moderate in the terminology introduced by Feichtinger and Gröbner in [2, Definition 3.1].

The next definition will be essential for the discretization problem.

Definition 5.2. A frame \mathcal{F} is said to possess property $D[\delta, m]$ if there exists a moderate admissible covering $\mathcal{U} = \mathcal{U}^\delta = (U_i)_{i \in I}$ of X and a [phase function](#) $\Gamma : X \times X \rightarrow \mathbb{C}$ with $|\Gamma| = 1$ such that (5.3) holds and such that the kernel $\text{osc}_{\mathcal{U}, \Gamma}$ defined by²

$$\text{osc}_{\mathcal{U}, \Gamma}(x, y) := \sup_{z \in Q_y} |\langle S^{-1} \psi_x, \psi_y - \Gamma(y, z) \psi_z \rangle| = \sup_{z \in Q_y} |R(x, y) - \Gamma(y, z) R(x, z)|,$$

¹Even for simple examples, e.g. the kernel associated to the short-time Fourier transform, cf. [9], there is strong evidence that the conditions derived in this chapter cannot be fulfilled by the kernel $\text{osc}_{\mathcal{U}}$ proposed in [4].

²See annotation 1.

where $Q_y := \bigcup_{i, y \in U_i} U_i$, satisfies

$$(5.4) \quad \|\text{osc}_{\mathcal{U}, \Gamma} \mathcal{A}_m\| < \delta.$$

We assume from now on that the frame \mathcal{F} possesses at least property $D[\delta, 1]$ for some $\delta > 0$. Furthermore, we only admit weight functions m (resp. spaces Y) for which the frame has the property $D[\delta, m]$ for some $\delta > 0$.

5.1. Preparations. Associated to a function space Y and to a moderate admissible covering $\mathcal{U} = (U_i)_{i \in I}$ we will define two sequence spaces. Before being able to state their definition we have to make sure that characteristic functions of compact sets are contained in Y .

Lemma 5.1. *If Q is an arbitrary compact subset of X then the characteristic function of Q is contained in Y .*

Proof. Assume that F is a non-zero function in Y . Then by solidity we may assume that F is positive. Clearly, there exists a non-zero continuous positive kernel $L \in \mathcal{A}_m$. The application of L to F yields a non-zero positive continuous function in Y (by the assumption on \mathcal{A}_m). Hence, there exists a compact set U with non-void interior such that $L(F)(x) > 0$ for all $x \in U$. By compactness of U and continuity of $L(F)$ there exists a constant C such that $\chi_U(x) \leq CL(F)(x)$ for all $x \in X$. By solidity χ_U is contained in Y . Now, we set $K(x, y) = \mu(U)^{-1} \chi_Q(x) \chi_U(y)$, which clearly is an element of \mathcal{A}_m by compactness of Q and U . It holds $\chi_Q = K(\chi_U)$ and hence $\chi_Q \in Y$. \square

Now we may define the spaces³

$$Y^b := Y^b(\mathcal{U}) := \{(\lambda_i)_{i \in I}, \|\sum_{i \in I} |\lambda_i| \chi_{U_i} |Y| < \infty\},$$

$$Y^h := Y^h(\mathcal{U}) := \{(\lambda_i)_{i \in I}, \|\sum_{i \in I} |\lambda_i| \mu(U_i)^{-1} \chi_{U_i} |Y| < \infty\}$$

with natural norms

$$\|(\lambda_i)_{i \in I} |Y^b\| := \|\sum_{i \in I} |\lambda_i| \chi_{U_i} |Y|\|,$$

$$\|(\lambda_i)_{i \in I} |Y^h\| := \|\sum_{i \in I} |\lambda_i| \mu(U_i)^{-1} \chi_{U_i} |Y|\|.$$

If the numbers $\mu(U_i)$ are bounded from above (by assumption they are bounded from below) then the two sequence spaces coincide. Lemma 5.1 implies that the finite sequences are contained in Y^b and Y^h . If the space $(Y, \|\cdot\| |Y|)$ is a solid Banach function space, then $(Y^b, \|\cdot\| |Y^b|)$ and $(Y^h, \|\cdot\| |Y^h|)$ are solid BK-spaces, i.e. solid Banach spaces of sequences for which convergence implies componentwise convergence (this can be seen, for example, as a consequence of Theorem 5.2(d) and the fact that $Y^b \subset Y^h$). Let us state some further properties of these spaces.

Theorem 5.2. (a) *The spaces $(Y^b, \|\cdot\| |Y^b|)$ and $(Y^h, \|\cdot\| |Y^h|)$ are Banach spaces.*

(b) *If the bounded functions with compact support are dense in Y , then the finite sequences are dense in Y^b and Y^h .*

³In [4], the spaces and natural norms do not fit together. This is corrected by exchanging λ_i for $|\lambda_i|$ in the sums. See also [11] for the correct definition.

- (c) Assume that w is a weight function on X such that its associated weight $m(x, y) = \max\{w(x)/w(y), w(y)/w(x)\}$ satisfies (5.3). For $Y = L_w^p(X, \mu)$, $1 \leq p \leq \infty$, it holds $Y^\flat = \ell_{b_p}^p(I)$ and $Y^\natural = \ell_{d_p}^p(I)$ with equivalent norms with

$$b_p(i) := \mu(U_i)^{1/p} \tilde{w}(i), \quad d_p(i) := \mu(U_i)^{1/p-1} \tilde{w}(i)$$

where $\tilde{w}(i) = \sup_{x \in U_i} w(x)$.

- (d) Suppose that (5.3) holds for the weight function m associated to Y and denote⁴ $\tilde{v}(i) = \sup_{x \in U_i} v(x)$ and $r(i) = \tilde{v}(i)\mu(U_i)$. Then Y^\natural is continuously embedded into $\ell_{1/r}^\infty(I)$.

Proof. The statements (a) and (b) are straightforward to prove.

For (c) note that $\left(\sum_{j=1}^N |x_j|\right)^p / \left(\sum_{j=1}^N |x_j|^p\right) \leq N^p$ for arbitrary $x_i \in \mathbb{C}$ and use the finite overlap property of $\mathcal{U} = (U_i)_{i \in I}$ to see that

$$\begin{aligned} \left\| \sum_{i \in I} |\lambda_i| \chi_{U_i} |L_w^p|^p \right\|^p &= \int_X \left| \sum_{i \in I} |\lambda_i| \chi_{U_i}(y) \right|^p w^p(y) d\mu(y) \\ &\leq N^p \sum_{i \in I} |\lambda_i|^p \int_{U_i} w^p(y) d\mu(y) \leq N^p \|(\lambda_i)_{i \in I}\|_{\ell_{b_p}^p}^p. \end{aligned}$$

Alternatively, invoke Lemma 5.4 below to obtain an estimate with a possibly different constant⁵. For the other direction

$$\begin{aligned} \|(\lambda_i)_{i \in I}\|_{\ell_{b_p}^p}^p &= \sum_{i \in I} |\lambda_i|^p \mu(U_i) \tilde{w}^p(i) \leq C_{m, \mathcal{U}}^p \sum_{i \in I} |\lambda_i|^p \left(\inf_{x \in U_i} w(x) \right)^p \int_X \chi_{U_i}^p(y) d\mu(y) \\ &\leq C_{m, \mathcal{U}}^p \int_X \sum_{i \in I} |\lambda_i|^p \chi_{U_i}^p(y) w^p(y) d\mu(y) \\ &\leq C_{m, \mathcal{U}}^p \int_X \left| \sum_{i \in I} |\lambda_i| \chi_{U_i}(y) \right|^p w^p(y) d\mu(y) = C_{m, \mathcal{U}}^p \left\| \sum_{i \in I} |\lambda_i| \chi_{U_i} |L_w^p|^p \right\|^p. \end{aligned}$$

Thus we have proven $(L_w^p)^\flat = \ell_{b_p}^p$. For $(L_w^p)^\natural = \ell_{d_p}^p$ just substitute $\mu(U_i)^{-1} \lambda_i$ for λ_i in the derivations above.

For (d) we need to show that $|\lambda_i| \leq Cr(i) \|(\lambda_j)_{j \in I}\|_{Y^\natural}$ for all $i \in I$. Fix some $k \in I$ and define the kernel

$$(5.5) \quad K_i(x, y) = \chi_{U_k}(x) \chi_{U_i}(y), \quad i \in I.$$

For any $i \in I$ we obtain

$$|\lambda_i| \chi_{U_k} = K_i(|\lambda_i| \mu(U_i)^{-1} \chi_{U_i}) \leq K_i\left(\sum_{j \in I} |\lambda_j| \mu(U_j)^{-1} \chi_{U_j}\right).$$

By solidity of Y we get

$$\begin{aligned} |\lambda_i| \|\chi_{U_k}\|_Y &\leq \|K_i\| \left\| \sum_{j \in I} |\lambda_j| \mu(U_j)^{-1} \chi_{U_j} \right\|_Y \leq \|K_i\|_{\mathcal{A}_m} \left\| \sum_{j \in I} |\lambda_j| \mu(U_j)^{-1} \chi_{U_j} \right\|_Y \\ &= \|K_i\|_{\mathcal{A}_m} \|(\lambda_j)_{j \in I}\|_{Y^\natural}. \end{aligned}$$

⁴Recall $v(x) = v_z(x) = m(x, z)$ for some fixed $z \in X$. Also recall that v_z and v_y are equivalent weights for all $z, y \in X$.

⁵ $\left\| \sum_{i \in I} |\lambda_i| \chi_{U_i} |L_w^p|^p \right\|^p \leq \sum_{i \in I} \int_{U_i} (\lambda_i^+)^p w^p(y) d\mu(y) \leq \|(\mu(U_i) \lambda_i^+)_{i \in I}\|_{Y^\natural} \leq \tilde{C} \|(\tilde{\lambda}_i^+)_{i \in I}\|_{Y^\natural} \leq \tilde{C} C \|(\lambda_i)_{i \in I}\|_{Y^\flat}$, where \tilde{C} is the moderateness constant from (5.2) and $\tilde{\lambda}_i = \mu(U_i) \lambda_i$.

Since k is fixed, it remains to show $\|K_i|_{\mathcal{A}_m}\| \leq Cr(i)$ to complete the proof. Let us estimate the \mathcal{A}_m -norm of K_i . With $y_0 \in U_k$ we obtain

$$\begin{aligned} \int_X |K_i(x, y)| m(x, y) d\mu(y) &\leq \chi_{U_k}(x) \int_{U_i} m(x, y) d\mu(y) \leq \mu(U_i) \sup_{x \in U_k} \sup_{y \in U_i} m(x, y) \\ &\leq \mu(U_i) \sup_{y \in U_i} m(y_0, y) \sup_{x \in U_k} m(x, y_0) \leq CC_{m, \mathcal{U}} \mu(U_i) \tilde{v}(i) = CC_{m, \mathcal{U}} r(i), \end{aligned}$$

where we used that m is admissible and that different choices of z in the definition (3.7) of v yield equivalent weights. Furthermore a similar computation yields

$$\begin{aligned} \int_X |K_i(x, y)| m(x, y) d\mu(x) &\leq \chi_{U_i}(y) \int_{U_k} m(x, y) d\mu(x) \leq CC_{m, \mathcal{U}} \mu(U_k) \tilde{v}(i) \\ &\leq CC_{m, \mathcal{U}} D^{-1} \mu(U_k) \mu(U_i) \tilde{v}(i) \end{aligned}$$

where D is the constant in Definition 5.1 of a moderate admissible covering and has been added to treat the case $\mu(U_i) < 1$. Hence, $\|K_i|_{\mathcal{A}_m}\| \leq C'r(i)$ for some suitable constant C' (note that k is fixed). This proves the claim. \square

Let us investigate the dependence of the spaces Y^\flat and Y^\natural on the particular covering chosen.

Definition 5.3. Suppose $\mathcal{U} = (U_i)_{i \in I}$ and $\mathcal{V} = (V_i)_{i \in I}$ are two moderate admissible coverings of X over the same index set I . Assume that m is a weight function on $X \times X$. The coverings \mathcal{U} and \mathcal{V} are called m -equivalent if the following conditions are satisfied.

- (i) There are constants $C_1, C_2 > 0$ such that $C_1 \mu(U_i) \leq \mu(V_i) \leq C_2 \mu(U_i)$ for all $i \in I$.
- (ii) There exists a constant C' such that $\sup_{x \in U_i} \sup_{y \in V_i} m(x, y) \leq C'$ for all $i \in I$.

Lemma 5.3. Let m be the weight function associated to Y and suppose that $\mathcal{U} = (U_i)_{i \in I}$ and $\mathcal{V} = (V_i)_{i \in I}$ are m -equivalent moderate admissible coverings over the same index set I . Then it holds $Y^\flat(\mathcal{U}) = Y^\flat(\mathcal{V})$ and $Y^\natural(\mathcal{U}) = Y^\natural(\mathcal{V})$ with equivalence of norms.

Proof. Assume that $(\lambda_i)_{i \in I}$ is contained in $Y^\flat(\mathcal{V})$. Observe that the term

$$\int_X \chi_{V_i}(y) \chi_{V_j}(y) d\mu(y) \mu(V_j)^{-1}$$

equals 1 for $i = j$ and for fixed i it is non-zero for at most N different indices j by the finite overlap property (5.1). We obtain

$$\begin{aligned} \sum_{i \in I} |\lambda_i| \chi_{U_i}(x) &\leq \sum_{i \in I} |\lambda_i| \sum_{j \in I} \chi_{U_j}(x) \int_X \chi_{V_i}(y) \chi_{V_j}(y) d\mu(y) \mu(V_j)^{-1} \\ &= \int_X \sum_{i \in I} |\lambda_i| \chi_{V_i}(y) \sum_{j \in I} \chi_{U_j}(x) \chi_{V_j}(y) \mu(V_j)^{-1} d\mu(y) = L\left(\sum_{i \in I} |\lambda_i| \chi_{V_i}\right)(x), \end{aligned}$$

where the kernel L is defined by

$$(5.6) \quad L(x, y) := \sum_{j \in I} \chi_{U_j}(x) \chi_{V_j}(y) \mu(V_j)^{-1}.$$

The interchange of summation and integration is always allowed since by the finite overlap property the sum is always finite for fixed x, y . We claim that L is contained in \mathcal{A}_m . Using property (ii) of

m -equivalent coverings and once more the finite overlap property, we get

$$\begin{aligned} \int_X L(x, y) m(x, y) d\mu(y) &= \sum_{j \in I} \chi_{U_j}(x) \int_X \chi_{V_j}(y) \mu(V_j)^{-1} m(x, y) d\mu(y) \\ &\stackrel{\text{(ii)}}{\leq} C' \sum_{j \in I} \chi_{U_j}(x) \leq C' N \quad \text{for all } x \in X. \end{aligned}$$

With property (i) and (ii) in Definition 5.3 we get

$$\begin{aligned} \int_X L(x, y) m(x, y) d\mu(x) &= \sum_{j \in I} \chi_{V_j}(y) \mu(V_j)^{-1} \int_X \chi_{U_j}(x) m(x, y) d\mu(x) \\ &\stackrel{\text{(ii)}}{\leq} C' \sum_{j \in I} \chi_{V_j}(y) \mu(V_j)^{-1} \mu(U_j) \stackrel{\text{(i)}}{\leq} C' C_1 N \quad \text{for all } y \in X. \end{aligned}$$

Thus, $L \in \mathcal{A}_m$ and by solidity of Y we conclude that

$$\|(\lambda_i)_{i \in I} |Y^b(\mathcal{U})\| \leq \|L(\sum_{i \in I} |\lambda_i| \chi_{V_i} |Y)\| \leq \|L|_{\mathcal{A}_m}\| \|(\lambda_i)_{i \in I} |Y^b(\mathcal{V})\|.$$

Exchanging the roles of \mathcal{U} and \mathcal{V} gives a reversed inequality and thus $Y^b(\mathcal{U}) = Y^b(\mathcal{V})$. Moreover, replacing $(\lambda_i)_{i \in I}$ by $(\mu(U_i)^{-1} \lambda_i)_{i \in I}$ shows that $Y^b(\mathcal{U}) = Y^b(\mathcal{V})$. \square

For some $i \in I$ we denote $i^* := \{j \in I, U_i \cap U_j \neq \emptyset\}$. Clearly, this is a finite set with at most N elements. The next Lemma states that the sequence spaces Y^b are \mathcal{U} -regular in the sense of [2, Definition 2.5].

Lemma 5.4. *For $(\lambda_i)_{i \in I} \in Y^b$ let $\lambda_i^+ := \sum_{j \in i^*} \lambda_j$. Then there exists some constant $C > 0$ such that $\|(\lambda_i^+)_{i \in I} |Y^b\| \leq C \|(\lambda_i)_{i \in I} |Y^b\|$.*

Proof. By Proposition 3.1 in [2] we have to prove that any permutation $\pi : I \rightarrow I$ satisfying $\pi(i) \in i^*$ for all $i \in I$ induces a bounded operator on Y^b , i.e. $\|(\lambda_{\pi(i)})_{i \in I} |Y^b\| \leq C' \|(\lambda_i)_{i \in I} |Y^b\|$, for some C' independent of π . We define the kernel

$$K_\pi(x, y) := \sum_{i \in I} \mu(U_{\pi^{-1}(i)})^{-1} \chi_{U_{\pi^{-1}(i)}}(x) \chi_{U_i}(y).$$

It is easy to see that

$$\begin{aligned} K_\pi(\mu(U_j)^{-1} \chi_{U_j})(x) &= \mu(U_{\pi^{-1}(j)})^{-1} \chi_{U_{\pi^{-1}(j)}}(x) + \sum_{i \in I \setminus \{j\}} \mu(U_{\pi^{-1}(i)})^{-1} \chi_{U_{\pi^{-1}(i)}}(x) \int_X \chi_{U_i}(y) \chi_{U_j}(y) d\mu(y) \\ &\geq \mu(U_{\pi^{-1}(j)})^{-1} \chi_{U_{\pi^{-1}(j)}}(x). \end{aligned}$$

This gives

$$\begin{aligned} \sum_{i \in I} |\lambda_{\pi(i)}| \mu(U_i)^{-1} \chi_{U_i}(x) &= \sum_{i \in I} |\lambda_i| \mu(U_{\pi^{-1}(i)})^{-1} \chi_{U_{\pi^{-1}(i)}}(x) \leq \sum_{i \in I} |\lambda_i| K_\pi(\mu(U_i)^{-1} \chi_{U_i})(x) \\ &= K_\pi(\sum_{i \in I} |\lambda_i| \mu(U_i)^{-1} \chi_{U_i})(x). \end{aligned}$$

Provided K_π is contained in \mathcal{A}_m this would give the result by solidity of Y . So let us estimate the \mathcal{A}_m -norm of K_π . We have

$$\begin{aligned} \int_X K_\pi(x, y) m(x, y) d\mu(x) &= \int_X \sum_{i \in I} \mu(U_{\pi^{-1}(i)})^{-1} \chi_{U_{\pi^{-1}(i)}}(x) \chi_{U_i}(y) m(x, y) d\mu(x) \\ &\leq \left(\sum_{i \in I} \chi_{U_i}(y) \right) \sup_{i \in I} \sup_{y \in U_i} \sup_{x \in \cup_{j \in i^*} U_j} m(x, y) \leq C_{m, \mathcal{U}}^2 N. \end{aligned}$$

Hereby, we used [the finite overlap property](#) and that for $y \in U_i, x \in U_j$ with $U_i \cap U_j \neq \emptyset$ and $z \in U_i \cap U_j$ it holds $m(x, y) \leq m(x, z) m(z, y) \leq C_{m, \mathcal{U}}^2$ by property (5.3). Furthermore by property (5.2), we obtain

$$\begin{aligned} \int_X K_\pi(x, y) m(x, y) d\mu(y) &= \int_X \sum_{i \in I} \mu(U_{\pi^{-1}(i)})^{-1} \chi_{U_{\pi^{-1}(i)}}(x) \chi_{U_i}(y) m(x, y) d\mu(y) \\ &\leq C_{m, \mathcal{U}}^2 \sum_{i \in I} \mu(U_{\pi^{-1}(i)})^{-1} \mu(U_i) \chi_{U_{\pi^{-1}(i)}}(x) \stackrel{(5.2)}{\leq} C_{m, \mathcal{U}}^2 \tilde{C} N. \end{aligned}$$

This completes the proof. \square

We will further need a partition of unity (PU) associated to a moderate admissible covering of X , i.e. a family $\Phi = (\phi_i)_{i \in I}$ of measurable functions that satisfies $0 \leq \phi_i(x) \leq 1$ for all $x \in X$, $\text{supp}(\phi_i) \subset U_i$ and $\sum_{i \in I} \phi_i(x) = 1$ for all $x \in X$. The construction of such a family Φ subordinate to a locally finite covering of some topological space is standard, see also [3, pp. 127-128, [Proposition 4.41](#)].

We may apply a kernel K also to a measure ν on X by means of

$$K(\nu)(x) = \int_X K(x, y) d\nu(y).$$

We define the following space of measures⁶,

$$D(\mathcal{U}, M, Y^\natural) := \{\nu \in M_{loc}(X), \|(|\nu|(U_i))_{i \in I}|Y^\natural\| < \infty\}$$

with norm

$$\|\nu|D(\mathcal{U}, M, Y^\natural)\| := \|(|\nu|(U_i))_{i \in I}|Y^\natural\|,$$

where M_{loc} denotes the space of complex Radon measures. Spaces of this kind were introduced by Feichtinger and Gröbner in [2] who called them decomposition spaces. We identify a function with a measure in the usual way. Then

$$D(\mathcal{U}, L^1, Y^\natural) := \{F \in L_{loc}^1, \|(\int_{U_i} |F(x)| d\mu(x))_{i \in I}|Y^\natural\| < \infty\}$$

with norm $\|F|D(\mathcal{U}, L^1, Y^\natural)\| := \|(\|\chi_{U_i} F|L^1\|)_{i \in I}|Y^\natural\|$ can be considered as a closed subspace of $D(\mathcal{U}, M, Y^\natural)$.

We have the following auxiliary result.

Lemma 5.5. (a) *It holds $Y \subset D(\mathcal{U}, L^1, (L_{1/v}^\infty)^\natural)$ with continuous embedding.*

⁶Our definition is formally slightly different, but clearly equivalent to the one in [4].

- (b) Assume that the frame \mathcal{F} has property $D[\delta, m]$ for some $\delta > 0$ and let $\mathcal{U}^\delta = (U_i)_{i \in I}$ be a corresponding⁷ moderate admissible covering of X . Then for $\nu \in D(\mathcal{U}^\delta, M, Y^\natural)$ it holds $R(\nu) \in Y$ and $\|R(\nu)|Y\| \leq C\|\nu\|D(\mathcal{U}^\delta, M, Y^\natural)\|$.

Proof. (a) Assume $F \in Y$ and let

$$H(x) := \sum_{i \in I} \|\chi_{U_i} F|L^1\| \mu(U_i)^{-1} \chi_{U_i}(x).$$

We need to prove $H \in L_{1/v}^\infty$. Fix $k \in I$. Since Y is continuously embedded into L_{loc}^1 by assumption⁸ there exists a constant C such that $\|\chi_{U_i} F|L^1\| \leq C\|F|Y\|$ for all $F \in Y$. With $K_i(x, y) = \chi_{U_k}(x)\chi_{U_i}(y)$ as in (5.5) (and fixed $k \in I$) it holds⁹ $\chi_{U_i} = \mu(U_k)^{-1} K_i^*(\chi_{U_k})$. It is shown in the proof of Theorem 5.2(d) that $\|K_i|\mathcal{A}_m\| \leq C'\mu(U_i)v(x_i)$ for some constant $C' > 0$ and some $x_i \in U_i$. We obtain¹⁰

$$\begin{aligned} \|\chi_{U_i} F|L^1\| &= \|\chi_{U_i}|F|L^1\| = \mu(U_k)^{-1} \|K_i^*(\chi_{U_k})|F|L^1\| = \mu(U_k)^{-1} \|\chi_{U_k} K_i(|F|)|L^1\| \\ &\leq C\mu(U_k)^{-1} \|K_i(|F|)|Y\| \leq C\mu(U_k)^{-1} \|K_i|\mathcal{A}_m\| \|F|Y\| = C''\mu(U_i)v(x_i) \|F|Y\| \\ &= C''\mu(U_i)v(x_i) \|F|Y\| \end{aligned}$$

where we used solidity of L^1 and Y . With this we obtain

$$H(x) \leq C''\|F|Y\| \sum_{i \in I} \chi_{U_i}(x)v(x_i).$$

For fixed x this is a finite sum over the index set $I_x = \{i \in I, x \in U_i\}$. It holds

$$\sup_{i \in I_x} v(x_i) \leq \sup_{i \in I_x} m(x_i, x)m(x, z) \leq C_{m, \mathcal{U}}m(x, z) = C_{m, \mathcal{U}}v(x)$$

by (5.3). This proves $H \in L_{1/v}^\infty$ and the embedding is continuous.

(b) Let $\Phi = (\phi_i)_{i \in I}$ be a PU associated to \mathcal{U} . Further, we denote $R_i(x, y) := \phi_i(y)R(x, y)$. Clearly we have $R(x, y) = \sum_{i \in I} R_i(x, y)$. We obtain

$$(A) \quad |R_i(\nu)(x)| = \left| \int_X R_i(x, y) d\nu(y) \right| \leq \int_{U_i} |R_i(x, y)| d|\nu|(y) \leq |\nu|(U_i) \|R_i(x, \cdot)\|_\infty.$$

On the other hand,

$$\mu(U_i) \|R_i(x, \cdot)\|_\infty \leq \int_X \chi_{U_i}(y) \sup_{z \in U_i} |R(x, z)| d\mu(y).$$

Since the frame \mathcal{F} is assumed to have property $D[\delta, m]$ we obtain by definition of $\text{osc}_{\mathcal{U}^\delta, \Gamma}$ that

$$\begin{aligned} |R(x, z)| &= |\Gamma(y, z)R(x, z)| = |R(x, y) + \Gamma(y, z)R(x, z) - R(x, y)| \\ &\leq |R(x, y)| + |R(x, y) - \Gamma(y, z)R(x, z)| \leq \text{osc}_{\mathcal{U}^\delta, \Gamma}(x, y) + |R(x, y)| \text{ for all } z, y \in U_i. \end{aligned}$$

This gives

$$(B) \quad \mu(U_i) \|R_i(x, \cdot)\|_\infty \leq \int_X \chi_{U_i}(y) (\text{osc}_{\mathcal{U}^\delta, \Gamma}(x, y) + |R(x, y)|) d\mu(y) = (\text{osc}_{\mathcal{U}^\delta, \Gamma} + |R|)(\chi_{U_i})(x).$$

⁷i.e. there is a $\Gamma : X \times X \rightarrow \mathbb{C}$ with $|\Gamma| = 1$ such that $\|\text{osc}_{\mathcal{U}^\delta, \Gamma}|\mathcal{A}_m\| < \delta$.

⁸See condition (Y1) in [4, Section 3].

⁹ $\mu(U_k)^{-1} K_i^*(\chi_{U_k})(x) = \mu(U_k)^{-1} \int_X \chi_{U_k}(y) \chi_{U_k}(y) \chi_{U_i}(x) d\mu(y) = \chi_{U_i}(x)$

¹⁰ $\|K_i^*(\chi_{U_k})|F|L^1\| = \langle K_i^*(\chi_{U_k}), |F| \rangle = \langle \chi_{U_k}, K_i(|F|) \rangle \|\chi_{U_k} K_i(|F|)|L^1\|$.

Combine (A) and (B) to find

$$\begin{aligned}
\|R(\nu)|Y\| &= \left\| \sum_{i \in I} R_i(\nu)|Y\| \right\| \leq \left\| \sum_{i \in I} |\nu|(U_i)\mu(U_i)^{-1}(\text{osc}_{\mathcal{U}^\delta, \Gamma} + |R|)(\chi_{U_i})|Y\| \right\| \\
&= \left\| (\text{osc}_{\mathcal{U}^\delta, \Gamma} + |R|) \left(\sum_{i \in I} |\nu|(U_i)\mu(U_i)^{-1}\chi_{U_i} \right) |Y\| \right\| \\
&\leq (\|\text{osc}_{\mathcal{U}^\delta, \Gamma}\mathcal{A}_m\| + \|R\mathcal{A}_m\|) \left\| \sum_{i \in I} |\nu|(U_i)\mu(U_i)^{-1}\chi_{U_i}|Y\| \right\| \\
(5.7) \quad &= (\|\text{osc}_{\mathcal{U}^\delta, \Gamma}\mathcal{A}_m\| + \|R\mathcal{A}_m\|) \|\nu|D(\mathcal{U}, M, Y^\natural)\|
\end{aligned}$$

where we used the finite overlap property of the covering \mathcal{U} . This proves the claim. \square

Using this Lemma we may prove that the assumption made in Proposition 3.7 holds in case that the general assumptions of this section are true¹¹.

Corollary 5.6. *If the frame \mathcal{F} has property $D[\delta, m]$ then $R(Y) \subset L_{1/v}^\infty$ with continuous embedding. In particular, Proposition 3.7 holds.*

Proof. Suppose $F \in Y$. By Lemma 5.5(a) it holds $F \in D(\mathcal{U}^\delta, L^1, (L_{1/v}^\infty)^\natural)$, where \mathcal{U}^δ is a covering of X such that $\|\text{osc}_{\mathcal{U}^\delta, \Gamma}\mathcal{A}_m\| < \delta$ for a suitable phase function Γ . Such \mathcal{U}^δ and Γ exist, since \mathcal{F} has property $D[\delta, m]$ and by Lemma 5.5(b) we get $R(F) \in L_{1/v}^\infty$. \square

5.2. Atomic Decompositions and Banach Frames. Let us give the definition of an atomic decomposition and of a Banach frame. For a Banach space B we denote its dual by B^* .

Definition 5.4. A family $(g_i)_{i \in I}$ in a Banach space is called an atomic decomposition for B if for some countable index set I there exist a BK-space $(B^\natural(I), \|\cdot\|_{B^\natural(I)})$, $B^\natural = B^\natural(I)$, and linear bounded functionals $(\lambda_i)_{i \in I} \subset B^*$ (not necessarily unique) such that

- $(\lambda_i(f))_{i \in I} \in B^\natural$ for all $f \in B$ and there exists a constant $0 < C_1 < \infty$ independent of f such that

$$\|(\lambda_i(f))_{i \in I}\|_{B^\natural} \leq C_1 \|f\|_B,$$

- if $(\lambda_i)_{i \in I} \in B^\natural$ then¹² $f = \sum_{i \in I} \lambda_i g_i \in B$ (with unconditional convergence in some suitable topology) and there exists a constant $0 < C_2 < \infty$ independent of $(\lambda_i)_{i \in I}$ such that

$$\|f\|_B \leq C_2 \|(\lambda_i)_{i \in I}\|_{B^\natural},$$

- $f = \sum_{i \in I} \lambda_i(f) g_i$ for all $f \in B$.

We remark that this is not a standard definition (and probably such is not available). For instance, Triebel uses the same terminology with a slightly different meaning [10, p.59 and p.160]. The next definition is due to Gröchenig [7].

Definition 5.5. Suppose $(B, \|\cdot\|_B)$ is a Banach space. A family $(h_i)_{i \in I} \subset B^*$ is called a Banach frame for B if there exists a BK-space $(B^\flat, \|\cdot\|_{B^\flat})$, $B^\flat = B^\flat(I)$, and a linear bounded reconstruction operator $\Omega : B^\flat \rightarrow B$ such that

¹¹i.e. \mathcal{F} possesses at least property $D[\delta, 1]$ for some $\delta > 0$ and Y is such that \mathcal{F} has property $D[\delta, m]$ for some $\delta > 0$, where m is the weight function associated to Y .

¹²Note that here $(\lambda_i)_{i \in I}$ is used to denote a sequence in the sequence space, not a sequence of functionals in B^* .

- if $f \in B$ then $((h_i(f)))_{i \in I} \in B^b$, and there exist constants $0 < C_1 \leq C_2 < \infty$ independent of f such that

$$C_1 \|f|B\| \leq \| (h_i(f))_{i \in I} |B^b \| \leq C_2 \|f|B\|,$$

- $\Omega(h_i(f))_{i \in I} = f$ for all $f \in B$.

Clearly, these definitions apply also with B^* replaced by the anti-dual B^\top . Now we are prepared to state the main result of this article.

Theorem 5.7. *Assume that m is an admissible weight. Suppose the frame $\mathcal{F} = \{\psi_x\}_{x \in X}$ possesses property $D[\delta, m]$ for some $\delta > 0$ and let \mathcal{U}^δ denote a corresponding moderate admissible covering of X such that*

$$(5.8) \quad \delta(\|R|\mathcal{A}_m\| + \max\{C_{m, \mathcal{U}^\delta} \|R|\mathcal{A}_m\|, \|R|\mathcal{A}_m\| + \delta\}) \leq 1$$

where $C_{m, \mathcal{U}^\delta}$ is the constant in (5.3). Choose points $(x_i)_{i \in I} \subset X$ such that $x_i \in U_i$. Moreover assume that $(Y, \|\cdot\|_Y)$ is a Banach space fulfilling properties (Y1) and (Y2).

Then $\mathcal{F}_d := \{\psi_{x_i}\}_{i \in I} \subset \mathcal{K}_v^1$ is both an atomic decomposition of $\widetilde{\text{CoY}}$ with corresponding sequence space Y^\natural and a Banach frame for CoY with corresponding sequence space Y^b . Moreover, there exists a 'dual frame' $\widehat{\mathcal{F}}_d := \{e_i\}_{i \in I} \subset \mathcal{H}_v^1$ such that

- (a) we have the norm equivalences

$$\|f| \text{CoY}\| \cong \|(\langle f, \psi_{x_i} \rangle)_{i \in I} |Y^b\| \quad \text{and} \quad \|f| \widetilde{\text{CoY}}\| \cong \|(\langle f, e_i \rangle)_{i \in I} |Y^\natural\|,$$

- (b) if $f \in \widetilde{\text{CoY}}$ then

$$f = \sum_{i \in I} \langle f, e_i \rangle \psi_{x_i}$$

with unconditional norm convergence in $\widetilde{\text{CoY}}$ if the finite sequences are dense in Y^\natural and with unconditional convergence in the weak-* topology induced from $(\mathcal{H}_v^1)^\top$ otherwise.

- (c) if the finite sequences are dense in Y^b , then for all $f \in \text{CoY}$ it holds

$$f = \sum_{i \in I} \langle f, \psi_{x_i} \rangle e_i$$

with unconditional convergence in the norm of CoY .

Also discretizations of the canonical dual frame lead to Banach frames and atomic decompositions.

Theorem 5.8. *Under the same assumptions and with the same notation as in the previous theorem $\widetilde{\mathcal{F}}_d := \{S^{-1}\psi_{x_i}\}_{i \in I} \subset \mathcal{H}_v^1$ is both an atomic decomposition of CoY (with corresponding sequence spaces Y^\natural) and a Banach frame for $\widetilde{\text{CoY}}$ (with corresponding sequence space Y^b). Moreover, there exists a 'dual frame' $\widehat{\widetilde{\mathcal{F}}}_d := \{\tilde{e}_i\}_{i \in I} \subset \mathcal{K}_v^1$ with the analogous properties as in the previous theorem.*

Let us remark that the two previous theorems hold “uniformly in Y ”. Namely, if m is fixed then the constant δ is the same for all function spaces Y satisfying properties (Y1) and (Y2) with that specific m . In particular, the same covering $\mathcal{U}^\delta = (U_i)_{i \in I}$ can be used for all those spaces Y and $(\psi_{x_i})_{i \in I}$, $x_i \in U_i$, is a Banach frame for all coorbit spaces CoY at the same time.

The previous theorems imply an embedding result.

Corollary 5.9. *We have the following continuous embeddings*

$$\mathcal{H}_v^1 \subset \text{CoY} \subset (\mathcal{K}_v^1)^\top \quad \text{and} \quad \mathcal{K}_v^1 \subset \widetilde{\text{CoY}} \subset (\mathcal{H}_v^1)^\top$$

Proof. By definition¹³ $f \in \widetilde{\text{CoY}}$ implies $f \in (\mathcal{H}_v^1)^\top$ and the embedding $\widetilde{\text{CoY}} \subset (\mathcal{H}_v^1)^\top$ is continuous by Lemma 3.2 and Corollary 5.6. Lemma 5.1 shows that the Dirac element $\delta_i(j) := \delta_{i,j}$ is contained in Y^\natural and this in turn implies with Theorem 5.7¹⁴ that all $\psi_{x_i}, i \in I$, are contained in $\widetilde{\text{CoY}}$ with¹⁵ $\|\psi_{x_i}|_{\widetilde{\text{CoY}}}\| \leq C_2 \|\delta_i|_{Y^\natural}\| = C_2 \|\mu(U_i)^{-1} \chi_{U_i}|_Y\| \leq CC_2 v(x_i)$. Since any $x \in X$ may be chosen as one of the x_i it holds $\psi_x \in \widetilde{\text{CoY}}$ for all $x \in X$ with $\|\psi_x|_{\widetilde{\text{CoY}}}\| \leq C' v(x)$. Corollary 3.4 hence implies that \mathcal{K}_v^1 is continuously embedded into $\widetilde{\text{CoY}}$. The other embeddings are shown analogously. \square

We will split the proof of Theorems 5.7 and 5.8 into several lemmas. Let us just explain shortly the idea. Given a moderate admissible covering $\mathcal{U}^\delta = (U_i)_{i \in I}$, a corresponding PU $(\phi_i)_{i \in I}$ and points $x_i \in U_i, i \in I$, we define the operator

$$(U) \quad U_\Phi F(x) := \sum_{i \in I} c_i F(x_i) R(x, x_i)$$

where $c_i = \int_X \phi_i(x) d\mu(x)$. Intuitively, U_Φ is a discretization of the integral operator R .

If U_Φ is close enough to the operator R on $R(Y)$ this implies that U_Φ is invertible on $R(Y)$ since R is the identity on $R(Y)$ by Proposition 3.7. Since $Wf \in R(Y)$ whenever $f \in \widetilde{\text{CoY}}$ and $R(x, x_i) = W(\psi_{x_i})(x)$ we conclude

$$Wf = U_\Phi U_\Phi^{-1} Wf = \sum_{i \in I} c_i (U_\Phi^{-1} Wf)(x_i) W(\psi_{x_i})$$

resulting in $f = \sum_{i \in I} c_i (U_\Phi^{-1} Wf)(x_i) \psi_{x_i}$ by the correspondence principle stated in Proposition 3.7, [once convergence is ensured](#). This is an expansion of an arbitrary $f \in \widetilde{\text{CoY}}$ into the elements $\psi_{x_i}, i \in I$, and thus it gives a strong hint that we have in fact an atomic decomposition. Reversing the order of U_Φ and U_Φ^{-1} and replacing Wf by Vf

$$Vf = U_\Phi^{-1} U_\Phi Vf = U_\Phi^{-1} \sum_{i \in I} c_i Vf(x_i) W(\psi_{x_i})$$

leads to a recovery of an arbitrary $f \in \text{CoY}$ from its coefficients $Vf(x_i) = \langle f, \psi_{x_i} \rangle$ and thus we may expect to have a Banach frame. In the following we will make this rough idea precise. In particular, we need to find conditions on δ that make sure that U_Φ is close enough to the identity on $R(Y)$ (in fact this is ensured by (5.8)). Moreover, we will need some results that enable us to prove corresponding norm equivalences.

Let us start with some technical lemmas.

Lemma 5.10. *Suppose that the frame \mathcal{F} has property $D[\delta, m]$ for some $\delta > 0$ and that $\mathcal{U}^\delta = (U_i)_{i \in I}$ is a corresponding moderate admissible covering of X . Further, assume $(\lambda_i)_{i \in I} \in Y^\natural$ and $(x_i)_{i \in I}$ to be points such that $x_i \in U_i$. Then $x \mapsto \sum_{i \in I} \lambda_i R(x, x_i)$ defines a function in Y and*

$$(5.9) \quad \left\| \sum_{i \in I} \lambda_i R(\cdot, x_i) \right\|_Y \leq C' \|(\lambda_i)_{i \in I}\|_{Y^\natural}.$$

¹³The reference to Proposition 3.7 and Corollary 5.6 in [4] to show $Wf \in R(Y) \in L_{1/v}^\infty$ is superfluous.

¹⁴By the second point of the definition of atomic decompositions.

¹⁵The reasoning behind the ℓ_v^1 estimate in [4] is unclear. Note however that we obtain, with K_i as in (5.5), $\chi_{U_i} = \mu(U_k)^{-1} K_i^*(\chi_{U_k})$. By the estimates in the proof of Theorem 5.2(d), we have $\|K_i|_{\mathcal{A}_m}\| \leq C_{m,\mathcal{U}} \max\{\mu(U_k), \mu(U_i)\} \sup_{x \in U_i} v(x)$. By the definition of v and property (5.3) of m , we obtain $\|\mu(U_i)^{-1} \chi_{U_i}|_Y\| \leq C_{m,\mathcal{U}}^2 D^{-1} \|\chi_{U_k}|_Y\| v(x_i)$ for all $i \in I$ and some fixed $k \in I$.

The convergence is pointwise, and if the finite sequences are dense in Y^\natural it is also in the norm of Y . Furthermore, the series $\sum_{i \in I} R(x, x_i) v(x_i)$ converges pointwise and absolutely to a function in $L_{1/v}^\infty$.

Proof. Denote by ϵ_x the Dirac measure in x . then the application of R to the measure $\nu := \sum_{i \in I} \lambda_i \epsilon_{x_i}$ results in the function $x \mapsto \sum_{i \in I} \lambda_i R(x, x_i)$. It follows from Lemma 5.4 that

$$\left\| \sum_{i \in I} \lambda_i \epsilon_{x_i} |D(\mathcal{U}^\delta, M, Y^\natural)| \right\| = \left\| \sum_{i \in I} |\nu|(U_i) |Y^\natural| \right\| \leq \|(|\lambda_i|^+)_{i \in I} |Y^\natural|\| \leq C \|(\lambda_i)_{i \in I} |Y^\natural|\|$$

where $|\lambda_i|^+ = \sum_{j \in i^*} |\lambda_j|$ and $i^* = \{j \in I, U_i \cap U_j \neq \emptyset\}$. Thus, Lemma 5.5(b) yields (5.9). If the finite sequences are dense in Y^\natural then clearly the convergence is in the norm of Y .

For the pointwise convergence in $L_{1/v}^\infty$ observe that the space $Y = L_{1/v}^\infty$ satisfies (Y2) with the associated weight function m . For this choice it holds $Y^\natural = \ell_{1/r}^\infty$ where $r(i) = v(x_i) \mu(U_i)$ (Theorem 5.2(c))¹⁶. The application of $|R|$ to the measure $\nu = \sum_{i \in I} v(x_i) \mu(U_i) \epsilon_{x_i}$ yields $\sum_{i \in I} |R(\cdot, x_i)| v(x_i) \mu(U_i)$. The estimations in (5.7) are also valid pointwise until the second line, yielding

$$R(\nu)(x) \leq (\text{osc}_{\mathcal{U}^\delta, \Gamma} + |R|) \left(\sum_{i \in I} |\nu|(U_i) \mu(U_i)^{-1} \chi_{U_i} \right)(x).$$

For our specific choice of ν we have

$$|\nu|(U_i) = \sum_{j, x_j \in U_i \cap U_j} |v(x_j)| \mu(U_j) \leq \sum_{j \in i^*} |v(x_j)| \mu(U_j) < \infty,$$

since this is a finite sum. Moreover, for fixed x also

$$H(x) = \sum_{i \in I} |\nu|(U_i) \mu(U_i)^{-1} \chi_{U_i}(x)$$

is a finite sum and hence converges pointwise. We already know that H is contained in $L_{1/v}^\infty$, since obviously $(v(x_i) \mu(U_i))_{i \in I} = (r(i))_{i \in I} \in \ell_{1/r}^\infty$. We conclude that the partial sums of $\sum_{i \in I} |R(x, x_i)| v(x_i) \mu(U_i)$ are dominated by

$$\begin{aligned} \int_X (\text{osc}_{\mathcal{U}^\delta, \Gamma} + |R|)(x, y) H(y) d\mu(y) &= \int_X (\text{osc}_{\mathcal{U}^\delta, \Gamma} + |R|)(x, y) v(y) H(y) v^{-1}(y) d\mu(y) \\ (5.10) \quad &\leq \int_X (\text{osc}_{\mathcal{U}^\delta, \Gamma} + |R|)(x, y) m(x, y) d\mu(y) m(x, z) \sup_{y \in X} (|H(y)| v^{-1}(y)) \\ &\leq m(x, z) \|\text{osc}_{\mathcal{U}^\delta, \Gamma} + |R|\|_{\mathcal{A}_m} \|H\|_{L_{1/v}^\infty} \end{aligned}$$

where we used the symmetry and property (3.2) of admissible weights. Hence, the sum $\sum_{i \in I} |R(\cdot, x_i)| v(x_i) \mu(U_i)$ converges pointwise. By Theorem 5.2(d) we have $Y^\natural \subset \ell_{1/r}^\infty$ with continuous embedding for general Y . Together with the results just proven this yields that the convergence is also pointwise in general. \square

¹⁶Note that $d_p(i) = (\tilde{v}(i) \mu(U_i))^{-1}$, where $\tilde{v}(i) = \inf_{x \in U_i} v(x)$, cp. Theorem 5.2. Furthermore, m is an admissible weight and hence $v(x) = m(x, z) \leq m(x, x_i) m(x_i, z) \leq C_{m, \mathcal{U}} v(x_i)$ and similarly $v(x_i) \leq C_{m, \mathcal{U}} v(x)$ for all $x \in U_i$ by (5.3). Therefore, $1/r(i)$ and $d_p(i)$ are equivalent weights.

Lemma 5.11. *Suppose that the frame \mathcal{F} has property $D[\delta, m]$ for some $\delta > 0$ and let $\mathcal{U}^\delta = (U_i)_{i \in I}$ be an associated moderate admissible covering of X with corresponding $PU(\phi_i)_{i \in I}$. If $F \in R(Y)$ then for some constant $D > 0$ it holds¹⁷*

$$(5.11) \quad \|(F(x_i))_{i \in I}|Y^\flat\| \leq D\|F|Y\| \quad \text{and} \quad \left\| \sum_{i \in I} |F(x_i)|\phi_i|Y\right\| \leq \sigma\|F|Y\|$$

where $\sigma := \max\{C_{m, \mathcal{U}^\delta}\|R|\mathcal{A}_m\|, \|R|\mathcal{A}_m\| + \delta\}$ with $C_{m, \mathcal{U}^\delta}$ being the constant in (5.3). In particular,

$$\left\| \sum_{i \in I} F(x_i)\chi_{U_i}|Y\right\| \leq D\|F|Y\| \quad \text{and} \quad \left\| \sum_{i \in I} F(x_i)\phi_i|Y\right\| \leq \sigma\|F|Y\|,$$

by solidity.

Proof. We prove (5.11), the last part of the lemma then follows by solidity of Y or more specifically $\|F|Y\| = \|F\|Y\|$. Since $F \in R(Y)$ it holds $F = R(F)$ by Proposition 3.7 and Corollary 5.6. This yields

$$\begin{aligned} H(x) &:= \sum_{i \in I} |F(x_i)|\chi_{U_i}(x) = \sum_{i \in I} |R(F)(x_i)|\chi_{U_i}(x) = \sum_{i \in I_x} \left| \int_X R(x_i, y)F(y)\chi_{U_i}(x)d\mu(y) \right| \\ &\leq \int_X \sum_{i \in I_x} |R(x_i, y)| |F(y)|\chi_{U_i}(x)d\mu(y). \end{aligned}$$

Since the sum is finite over the index set $I_x = \{i \in I, x \in U_i\}$ the interchange of summation and integration is justified. Define

$$(5.12) \quad K(x, y) := \sum_{i \in I} |R(x_i, y)|\chi_{U_i}(x)$$

we obtain $H \leq K(|F|)$. We claim that $K \in \mathcal{A}_m$. For the integral with respect to y we obtain

$$\int_X |K(x, y)|m(x, y)d\mu(y) \leq \sum_{i \in I_x} \chi_{U_i}(x)m(x, x_i) \int_X |R(x_i, y)|m(x_i, y)d\mu(y) \leq NC_{m, \mathcal{U}^\delta}\|R|\mathcal{A}_m\|$$

where we used property (3.2) of m and N is the constant from (5.1). For an estimation of the integral with respect to x observe first that

$$\begin{aligned} |R(x_i, y)| &\leq |\overline{\Gamma}(x, x_i)R(x_i, y) - R(x, y)| + |R(x, y)| \\ &= |\Gamma(x, x_i)R(y, x_i) - R(y, x)| + |R(x, y)| \\ &\leq \text{osc}_{\mathcal{U}^\delta, \Gamma}(y, x) + |R(x, y)| = \text{osc}_{\mathcal{U}^\delta, \Gamma}^*(x, y) + |R(x, y)|, \end{aligned}$$

for all $x \in Q_{x_i} = \bigcup_{j, U_i \cap U_j \neq \emptyset} U_j$. By Fubini's theorem we obtain

$$\begin{aligned} \int_X |K(x, y)|m(x, y)d\mu(x) &= \int_X \sum_{i \in I} \chi_{U_i}(x)|R(x_i, y)|m(x, y)d\mu(x) \\ &\leq \sum_{i \in I} \int_{U_i} (\text{osc}_{\mathcal{U}^\delta, \Gamma}^*(x, y) + |R(x, y)|)m(x, y)d\mu(x) \leq N \int_X (\text{osc}_{\mathcal{U}^\delta, \Gamma}^*(x, y) + |R(x, y)|)m(x, y)d\mu(x) \\ &\leq N(\|\text{osc}_{\mathcal{U}^\delta, \Gamma}^*|\mathcal{A}_m\| + \|R|\mathcal{A}_m\|) < N(\|R|\mathcal{A}_m\| + \delta), \end{aligned}$$

¹⁷We state the lemma in this modified, stronger form, since our form of (5.11) is in fact invoked in both Corollary 5.12 and Theorem 5.13, cp. [4]. Note that the proof only requires trivial modification.

since clearly $\|\text{osc}_{\mathcal{U}^\delta, \Gamma}^*|\mathcal{A}_m\| = \|\text{osc}_{\mathcal{U}^\delta, \Gamma}|\mathcal{A}_m\|$. This proves $K \in \mathcal{A}_m$ and we finally obtain

$$\|(F(x_i))_{i \in I}|Y^\flat\| = \left\| \sum_{i \in I} |F(x_i)|\chi_{U_i}|Y\right\| \leq \|K(|F|)|Y\| \leq \|K|\mathcal{A}_m\| \|F\| \|Y\| = \|K|\mathcal{A}_m\| \|F\| \|Y\|.$$

A similar analysis shows also the second inequality in (5.11). The constant N from (5.1) does not enter the number σ since we replace the characteristic functions by a partition of unity. \square

Corollary 5.12. *Suppose the frame \mathcal{F} possesses property $D[\delta, m]$ for some $\delta > 0$. If $f \in \text{Co}Y$ then it holds $\|(Vf(x_i))_{i \in I}|Y^\flat\| \leq D\|f| \text{Co}Y\|$, where D is the constant in (5.11).*

Proof. By Proposition 3.7 it holds $Vf \in R(Y)$. By Lemma 5.11 we conclude $\|(Vf(x_i))_{i \in I}|Y^\flat\| \leq D\|Vf|Y\| = D\|f| \text{Co}Y\|$. \square

As already announced we need to show that U_Φ is invertible if δ is small enough.

Theorem 5.13. *Suppose the frame \mathcal{F} possesses property $D[\delta, m]$ for some $\delta > 0$ and let $\mathcal{U}^\delta = (U_i)_{i \in I}$ be an associated moderate admissible covering of X with corresponding $PU(\phi_i)_{i \in I}$. Let U_Φ be as in (U). Then it holds*

$$(5.13) \quad \|(\text{Id} - U_\Phi)|R(Y) \rightarrow R(Y)\| \leq \delta(\|R|\mathcal{A}_m\| + \sigma),$$

where σ is the constant from Lemma 5.11. Consequently, U_Φ is bounded and if the right hand side of (5.13) is less or equal to 1 the U_Φ is boundedly invertible on $R(Y)$.

Proof. Let us first show the implicit assertion that $F \in R(Y)$ implies $U_\Phi(F) \in R(Y)$. By Lemma 5.11 $(F(x_i))_{i \in I} \in Y^\flat$ which implies $(c_i F(x_i))_{i \in I} \in Y^\flat$. It follows from Lemma 5.10 that $\sum_{i \in I} c_i F(x_i)R(\cdot, x_i)$ converges pointwise to a function $G = U_\Phi(F) \in Y$. The pointwise convergence implies the weak-* convergence of $\sum_{i \in I} c_i F(x_i)\psi_{x_i}$ to an element g of $(\mathcal{H}_v^1)^\top$ by Lemma 3.6(b) which is then automatically contained in $\widetilde{\text{Co}Y}$ since $G \in Y$. From Lemma 3.6(c) follows that $G = Wg = R(Wg)$ and hence $U_\Phi(F) \in R(Y)$, implying $R(U_\Phi(F)) = U_\Phi(F)$.

Let us now introduce the auxiliary operator¹⁸

$$S_\Phi F(x) := R\left(\sum_{i \in I} \overline{\Gamma(\cdot, x_i)} F(x_i) \phi_i\right)(x),$$

where Γ is the phase function from the definition of $\text{osc}_{\mathcal{U}, \Gamma}$. By the triangle inequality,

$$(T) \quad \|F - U_\Phi F|Y\| \leq \|F - S_\Phi F|Y\| + \|S_\Phi F - U_\Phi F|Y\|.$$

We now estimate both terms on the RHS separately. Assuming $F \in R(Y)$ implies $F = R(F)$ by Proposition 3.7 and Corollary 5.6. This yields

$$\begin{aligned} \|F - S_\Phi F|Y\| &= \|R(F - \sum_{i \in I} \overline{\Gamma(\cdot, x_i)} F(x_i) \phi_i)|Y\| \\ &\leq \|R|\mathcal{A}_m\| \left\| \sum_{i \in I} \left(F - \overline{\Gamma(\cdot, x_i)} F(x_i)\right) \phi_i |Y\right\|. \end{aligned}$$

¹⁸The rest of the proof has to be adjusted to take into account the modified definition of $\text{osc}_{\mathcal{U}, \Gamma}$.

In order to estimate $\| (F - \overline{\Gamma(\cdot, x_i)} F(x_i)) \phi_i | Y \|$, examine

$$\begin{aligned}
& \left| \sum_{i \in I} \left(F(x) - \overline{\Gamma(x, x_i)} F(x_i) \right) \phi_i(x) \right| = \left| \sum_{i \in I} \left(R(F)(x) - \overline{\Gamma(x, x_i)} R(F)(x_i) \right) \phi_i(x) \right| \\
& = \left| \sum_{i \in I} \int_X F(y) \left(R(x, y) - \overline{\Gamma(x, x_i)} R(x_i, y) \right) d\mu(y) \phi_i(x) \right| \\
& \leq \sum_{i \in I} \int_X |F(y)| |R(x, y) - \Gamma(x, x_i) R(y, x_i)| d\mu(y) \phi_i(x) \\
& \leq \sum_{i \in I} \int_X |F(y)| \text{osc}_{\mathcal{U}^\delta, \Gamma}(y, x) d\mu(y) \phi_i(x) \\
& = \sum_{i \in I} \text{osc}_{\mathcal{U}^\delta, \Gamma}^*(|F|)(x) \phi_i(x) = \text{osc}_{\mathcal{U}^\delta, \Gamma}^*(|F|)(x).
\end{aligned}$$

In the derivations above, we used $R(x, y) = \overline{R(y, x)}$ and the property $\text{supp}(\phi_i) \subseteq U_i^\delta \in \mathcal{U}$ of the PU $\Phi = (\phi_i)_{i \in I}$. Furthermore, the interchange of summation and integration in the last line is allowed since by (5.1) the sum is finite for any fixed $x \in X$.

We obtain

$$(5.14) \quad \|F - S_\Phi F|Y\| \leq \|R| \mathcal{A}_m\| \|\text{osc}_{\mathcal{U}^\delta, \Gamma} | \mathcal{A}_m\| \|F|Y\|,$$

since $\|\text{osc}_{\mathcal{U}^\delta, \Gamma}^* | \mathcal{A}_m\| = \|\text{osc}_{\mathcal{U}^\delta, \Gamma} | \mathcal{A}_m\|$.

Now, we estimate $\|S_\Phi F - U_\Phi F|Y\|$. Note that

$$\begin{aligned}
& |S_\Phi(F)(x) - U_\Phi(F)(x)| \\
& = \left| \sum_{i \in I} \int_X F(x_i) \phi_i(y) \left(\overline{\Gamma(y, x_i)} R(x, y) - R(x, x_i) \right) d\mu(y) \right| \\
& \leq \sum_{i \in I} \int_X |F(x_i)| \phi_i(y) |R(x, y) - \Gamma(y, x_i) R(x, x_i)| d\mu(y) \\
& \leq \sum_{i \in I} \int_X |F(x_i)| \phi_i(y) \text{osc}_{\mathcal{U}^\delta, \Gamma}(x, y) d\mu(y),
\end{aligned}$$

where we used $\text{supp}(\phi_i) \subseteq U_i \in \mathcal{U}^\delta$ once more.

Define $H(y) := \sum_{i \in I} |F(x_i)| \phi_i(y)$, then by Lemma 5.11 and solidity of Y :

$$\begin{aligned}
(5.15) \quad \|S_\Phi F - U_\Phi F|Y\| & \leq \|\text{osc}_{\mathcal{U}^\delta, \Gamma} | \mathcal{A}_m\| \|H|Y\| \\
& \leq \sigma \|\text{osc}_{\mathcal{U}^\delta, \Gamma} | \mathcal{A}_m\| \|F|Y\|.
\end{aligned}$$

Insert (5.14) and (5.15) into (T) and use $\|\text{osc}_{\mathcal{U}^\delta, \Gamma} | \mathcal{A}_m\| < \delta$ to complete the proof. \square

Now we have all the ingredients to prove Theorem 5.7.

Proof of Theorem 5.7. The condition on δ implies by Theorem 5.13 that U_Φ is invertible on $R(Y)$. Assuming $f \in \widetilde{\text{Co}}Y$ means $Wf \in R(Y)$ by Lemma 3.6(c) and¹⁹ the definition of $\widetilde{\text{Co}}Y$. We conclude

$$Wf(x) = U_\Phi U_\Phi^{-1} Wf(x) = \sum_{i \in I} c_i (U_\Phi^{-1} Wf)(x_i) R(x, x_i) = \sum_{i \in I} c_i (U_\Phi^{-1} Wf)(x_i) W\psi_{x_i}(x).$$

Setting $\lambda_i(f) := c_i (U_\Phi^{-1} Wf)(x_i)$ we obtain with Proposition 3.7(b) and Corollary 5.6

$$(5.16) \quad f = \sum_{i \in I} \lambda_i(f) \psi_{x_i}.$$

Since $c_i \leq \mu(U_i)$ we obtain²⁰ with Lemma 5.11

$$\begin{aligned} \|(\lambda_i(f))_{i \in I}\|_{Y^\natural} &\leq \|((U_\Phi^{-1} Wf)(x_i))_{i \in I}\|_{Y^\natural} \leq C \|U_\Phi^{-1} Wf\|_Y \\ &\leq C \|U_\Phi^{-1}\|_{R(Y)} \|f\|_{\widetilde{\text{Co}}Y}, \end{aligned}$$

where we used $Wf \in R(Y)$ again. Conversely, suppose that $(\lambda_i)_{i \in I} \in Y^\natural$ and form the function

$$H(x) := \sum_{i \in I} \lambda_i R(x, x_i) = \sum_{i \in I} \lambda_i W\psi_{x_i}(x).$$

Since $Y^\natural \subset \ell_{1/r}^\infty$ (Theorem 5.2(d)) the sum converges pointwise to a function²¹ in $L_{1/v}^\infty$ by Lemma 5.10. By Lemma 3.6(b) the pointwise convergence of the partial sums of H implies the weak-* convergence in $(\mathcal{H}_v^1)^\natural$ of $f := \sum_{i \in I} \lambda_i \psi_{x_i}$. Hence, f is an element of $(\mathcal{H}_v^1)^\natural$ and by Lemma 5.10 is therefore contained in $\widetilde{\text{Co}}Y$. Also from Lemma 5.10 follows

$$\|f\|_{\widetilde{\text{Co}}Y} = \|H\|_Y \leq C' \|(\lambda_i)_{i \in I}\|_{Y^\natural}$$

and the convergence of the sum representing f is in the norm of $\widetilde{\text{Co}}Y$ if the finite sequences are dense in Y^\natural . This proves that $\mathcal{F}_d = (\psi_{x_i})_{i \in I}$ is an atomic decomposition of $\widetilde{\text{Co}}Y$.

Now suppose $f \in \text{Co}Y$ and let $F := Vf \in R(Y)$. We obtain

$$(5.17) \quad Vf = U_\Phi^{-1} U_\Phi Vf = U_\Phi^{-1} \left(\sum_{i \in I} c_i Vf(x_i) W\psi_{x_i} \right).$$

By the correspondence principle (Proposition 3.7) this implies

$$f = V^{-1} U_\Phi^{-1} \left(\sum_{i \in I} c_i Vf(x_i) R(\cdot, x_i) \right).$$

This is a reconstruction of f from the coefficients $Vf(x_i) = \langle f, \psi_{x_i} \rangle$, $i \in I$, and the reconstruction operator $T : Y^\natural \rightarrow \text{Co}Y$, $T = V^{-1} U_\Phi^{-1} J$ is bounded as the composition of bounded operators. Note that the operator²² $J((\lambda_i)_{i \in I})(x) := \sum_{i \in I} c_i \lambda_i R(x, x_i)$ is bounded by Lemma 5.10. Setting $Y = L_{1/v}^\infty$ shows that any element of $\text{Co}L_{1/v}^\infty = (\mathcal{K}_v^1)^\natural$ can be reconstructed in this way. Now, if for $f \in (\mathcal{K}_v^1)^\natural$ it holds $(\langle f, \psi_{x_i} \rangle)_{i \in I} \in Y^\natural$ then the series $\sum_{i \in I} \langle f, \psi_{x_i} \rangle \phi_i$ converges to an element of Y

¹⁹The original proof in [4] refers to Proposition 3.7(a) and Corollary 5.6, which seems does not seem to be required here. However, Proposition 3.7(b) and Corollary 5.6 also imply $Wf \in R(Y)$ and will be used again in this context, see (5.16).

²⁰Recall that $\|\cdot\|_B$ denotes the operator norm in $\mathcal{B}(B)$.

²¹Use Theorem 5.2(c).

²²Note $(\lambda_i)_{i \in I} \in Y^\natural$ implies $(c_i \lambda_i)_{i \in I} \in Y^\natural$.

since $\phi_i \leq \chi_{U_i}$. By bounded invertibility of U_Φ on $R(Y)$ the right hand side of (5.17) defines an element in Y , hence $f \in \text{Co}Y$.

Using (5.17), the norm equivalence follows from

$$\begin{aligned} \|f|_{\text{Co}Y}\| &= \|Vf|_Y\| \leq \|U_\Phi^{-1}|_{R(Y)}\| \left\| \sum_{i \in I} c_i V f(x_i) R(\cdot, x_i) |Y \right\| \\ &\leq C \|U_\Phi^{-1}\| \|(c_i V f(x_i))_{i \in I}\|_{Y^\natural} \leq C \|U_\Phi^{-1}\| \|(V f(x_i))_{i \in I}\|_{Y^\natural} \leq C' \|f|_{\text{Co}Y}\|. \end{aligned}$$

Hereby, we used Lemma 5.10, $c_i \leq \mu(U_i)$ and Corollary 5.12. Hence, we showed that \mathcal{F}_d is a Banach frame for $\text{Co}Y$.

In order to prove the existence of a dual frame let $E_i := c_i U_\Phi^{-1}(W\psi_{x_i}) \in R(L_v^1)$ and denote $e_i \in \mathcal{H}_v^1$ the unique vector such that $E_i = V(e_i)$. If the finite sequences are dense in Y^\natural then we may conclude from (5.17) by a standard argument (see also²³ [8, Lemma 5.3.2]) that $f = \sum_{i \in I} \langle f, \psi_{x_i} \rangle e_i$ with [unconditional](#) norm convergence. This proves (c).

We claim that

$$\lambda_i(f) = \langle f, e_i \rangle$$

yielding together with (5.16) $f = \sum_{i \in I} \langle f, e_i \rangle \psi_{x_i}$ (with weak-* convergence in general, and if the finite sequences are dense in Y^\natural with norm convergence).

If $F \in R(Y)$ then $F(x) = R(F)(x) = \langle F, W\psi_x \rangle$, by [self-adjointness of \$S^{-1}\$](#) . A simple calculation shows²⁴

$$\langle U_\Phi F, W\psi_x \rangle = \sum_{i \in I} c_i F(x_i) \langle R(\cdot, x_i), W\psi_x \rangle = \sum_{i \in I} c_i F(x_i) \overline{W\psi_x(x_i)} = \langle F, U_\Phi W\psi_x \rangle.$$

Hence, the same relation applies to $U_\Phi^{-1} = \sum_{i \in I} \sum_{n=0}^{\infty} (\text{Id} - U_\Phi)^n$ and we obtain

$$\begin{aligned} \lambda_i(f) &= c_i (U_\Phi^{-1} W f)(x_i) = c_i \langle U_\Phi^{-1} W f, W\psi_{x_i} \rangle = \langle W f, c_i U_\Phi^{-1} W\psi_{x_i} \rangle \\ &= \langle W f, V e_i \rangle = \langle f, W^* V e_i \rangle = \langle f, e_i \rangle. \end{aligned}$$

[Note that \$W^* = V^{-1}\$ on \$\mathcal{H}_v^1 \subseteq \mathcal{H}\$](#) . By Lemma 5.10 and 5.11 we have the norm estimate

$$\begin{aligned} \|f|_{\text{Co}Y}\| &= \left\| \sum_{i \in I} \langle f, e_i \rangle R(\cdot, x_i) |Y \right\| \leq C \|(\langle f, e_i \rangle)_{i \in I}\|_{Y^\natural} \leq C \|(U_\Phi^{-1} W f(x_i))_{i \in I}\|_{Y^\natural} \\ &\leq C' \|U_\Phi^{-1} W f|_Y\| \leq C' \|U_\Phi^{-1}\| \|f|_{\text{Co}Y}\|. \end{aligned}$$

This shows (a) and thus we complete the proof of Theorem 5.7. Theorem 5.8 is proved in the same way by exchanging the roles of V and W . \square

Remark 5.1. Using different approximation operators (compare^[7]²⁵) one can prove that under some weaker condition on δ one may discretize the continuous frame in order to obtain only atomic decompositions or only Banach frames with no corresponding results about (discrete) dual frames. In particular, if $\delta \leq 1$ then with the procedure of Theorem 5.7 one obtains atomic decompositions and in $\delta \leq \|R|_{\mathcal{A}_m}\|^{-1}$ one obtains Banach frames.

²³The original reference here is to Lemma 4, which does not exist in [8].

²⁴We have $\langle R(\cdot, x_i), W\psi_x \rangle = \overline{\langle W\psi_x, R(\cdot, x_i) \rangle} = \overline{W\psi_x(x_i)}$ and $\overline{W\psi_x(x_i) F(x_i)} = \overline{W\psi_x(x_i)} R(F)(x_i) = \langle F, U_\Phi W\psi_x \rangle$.

²⁵The original reference here is to [8] which does not seem to treat the discretization problem in detail.

Let us also add some comments about the Hilbert spaces situation which was the original question of Ali, Antoine and Gazeau. Here, we need to consider $Y = L^2$ since $\text{Co}L^2 = \widetilde{\text{Co}}L^2 = \mathcal{H}$. By [Theorem 5.2\(c\)](#) the corresponding sequence space is $Y^b = \ell^2_{\sqrt{a}}(I) = \ell^2(I, a)$ where $a_i = \mu(U_i)$. In order to be consistent with the usual notation of a (discrete) frame it seems suitable to renormalize the frame, i.e. under the conditions stated in [Theorem 5.7](#) (according to [Remark 5.1](#) it is only necessary to have $\delta \leq \|R|\mathcal{A}_m\|^{-1}$) it holds

$$C_1 \|f|\mathcal{H}\|^2 \leq \sum_{i \in I} |\langle f, \mu(U_i)^{1/2} \psi_{x_i} \rangle|^2 \leq C_2 \|f|\mathcal{H}\|^2.$$

This means that $(\mu(U_i)^{1/2} \psi_{x_i})_{i \in I}$ is a (Hilbert) frame in the usual sense. Of course, for the aim of Hilbert frames one may choose the trivial weight $m = 1$ in [Theorem 5.7](#).

One might ask whether the L^1 -integrability condition $R \in \mathcal{A}_1$ is necessary in order to obtain a Hilbert frame by discretizing the continuous frame. The crucial point in the proof of [Theorem 5.7](#) is that the operator U_Φ satisfies

$$(5.18) \quad \|U_\Phi - \text{Id}|V(\mathcal{H}) \rightarrow V(\mathcal{H})\| < 1.$$

If one finds a method to prove this without using integrability assumptions on R then the rest of the proof of [Theorem 5.7](#) should still work. However, it is not clear to us how to do this in general.

Concerning a complementary result Führ gave the example of a continuous frame indexed by \mathbb{R} which does not admit a discretization by any regular grid of \mathbb{R} [[5](#), [Example 1.6.9](#)]²⁶.

[This closes²⁷ Section 5.](#)

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²⁶The referenced result is not easily accessible using standard sources. However, the example referred to seems to be reproduced in [[6](#), [Example 2.36](#)]

²⁷We decided to omit [Remark 5.2](#) from [[4](#)] since sufficiency of the stated assumptions for a generalization of the discretization results to the setting of [Remark 3.2](#) requires step-by-step confirmation. Such an endeavor is beyond the scope of this annotation.

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